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TRAVELING PULSE SOLUTIONS IN A POINT MASS MODEL OF DIFFUSING PARTICLES

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Abstract

We study a partial differential equation modeling the self-motion of camphor parti- cles atop a water surface. The model equation is presented in the form of a reaction-diffusion system where source terms are expressed by delta functions. The resulting system is a point mass model for diffusing particles, where the role of the delta functions is to express camphor source locations. In our model, point sources interact with each other and move by the gradi- ent of the concentration field. We will discuss analytical properties of the model equation. In particular, we will study properties of traveling pulse solutions, whose existence are reduced to the solution of an ordinary differential equation, coupled with a boundary value problem. The existence and stability of solutions will be shown and we will compare our findings with those which have utilized characteristic function source terms.

Key words: camphor model; dirac delta function; self-motion

1. INTRODUCTION

Camphor has a surface activity and is therefore able to decreases the surface tension of water. Thus, when placed atop a water surface, inhomogeneity in surface tension induces its motion. We study the self-propelled motion of camphor particles atop a water channel using experimental, analytic, and computational techniques. Whereas previous studies have modeled the location of the camphor and its corresponding source term in the bulk equation using charac- teristic functions (see, for example Heisler et al., 2012; Nakata et al., 2001), a main goal of this research is to illustrate and compare the usage of delta function as source terms.

Regarding the mathematical modeling of the camphor's motion, since the bulk equation represents the camphor concentration, characteristic function source terms lead to the formulation of a moving boundary problem (where the concentration is constant at the location of the camphor). The precise treatment of such a model equation is technical, even in the one dimensional setting, and this aspect is further complicated in the two dimensional setting. This is particularly true when the shape of the camphor is allowed to vary. On the other hand, the delta function approach employed here allows us to sidestep such issues by treating the location of the camphor as a single point and it is thus meaningful and interesting to compare the two approaches.

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The outline of the paper is as follows. We begin with a discussion regarding experimental observations involving the self-driven motion of camphor disks atop a circular water channel. We then explain model equations which have been used in previous studies to model the observed phenomena. We modify these equations in order to treat the camphor as a point-mass, by means of delta functions. In order to compare features of our model with those of previous studies, we then construct traveling pulse solutions to our model equations. Computational and analytic techniques are then used to study the stability of these solutions, which are found to agree with results obtained under characteristic function source terms (Nagayama et al., 2004).

2. EXPERIMENTAL OBSERVATIONS

A camphor disc with a diameter of approximately 3 mm and a thickness of approximately 1 mm was placed atop a water chamber and its motion was observed. Although the camphor continues to move across the water surface well longer, the data used in this study correspond to approximately 25 s from the time the camphor was places atop the water channel. A total of 750 images were captured at a 30 frames per second and the camphor used in our experiments was obtained from Wako Pure Chemical Industries.

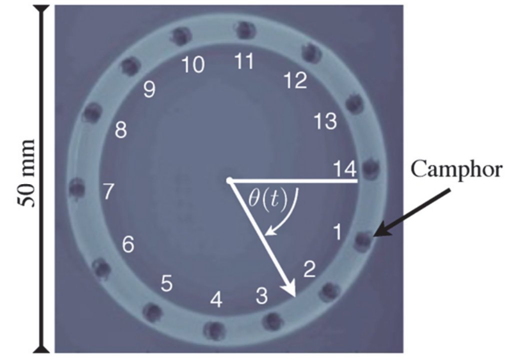

Fig. 1. Camphor disk motion. The location of the camphor disk along the water channel is superimposed using an approximate time interval of 0.17 s. The snapshots corresponds to the camphor's first traversal (within the time interval [0, 2.27] s), where they are labeled with respect to time.

Figure 1 illustrates the camphor's motion during the first 2.27 s. Starting from location 1, the camphor moves in a clockwise direction towards location 2. As shown in the figure, the distance traveled by the camphor between each time interval is approximately constant. This observation is measured in figure 2, where we have manually approximated the center of mass of the camphor disk at each of the 750 frames. We observe that the slope of $\theta(t)$ is approx- imately constant, and a closer inspection (figure 3) shows the angular frequency fluctuating around a median value of 2.53 rad/s. Although the angular velocity of the camphor's motion is not con-

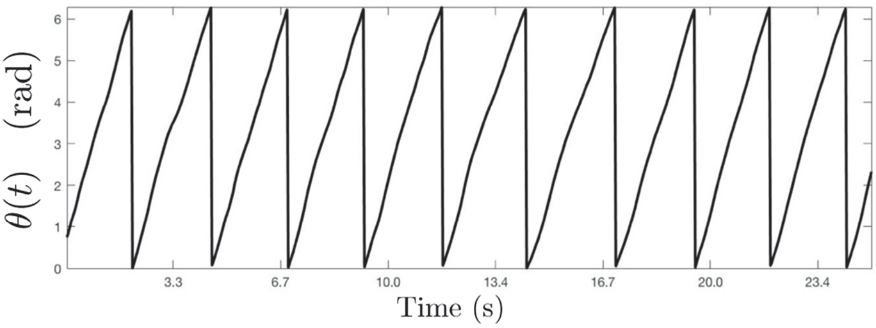

Fig. 2. Time evolution of θ(t), as defined in figure 1. The camphor exhibits approximately constant angular velocity (the abrupt change in angle corresponds to the 2π periodicity).

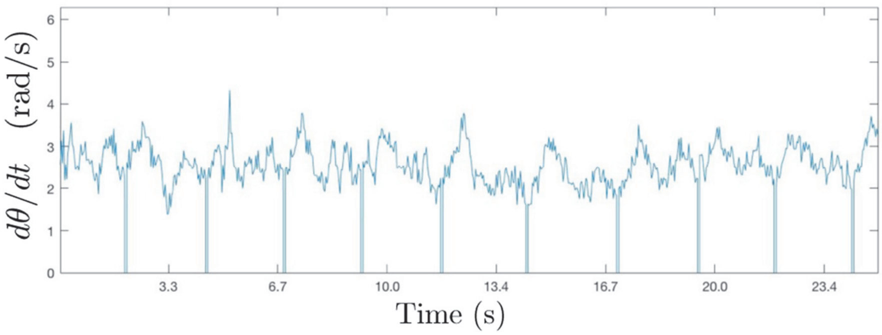

Fig. 3. Approximation of the camphor's observed angular velocity. The jumps correspond to the 2π periodicity of the experimental setting.

stant, we remark that much of the noise can be attributed to our approximation of the camphor's centre.

3. MATHEMATICAL MODELING OF CAMPHOR MOTION

Previous studies model the dynamics of the camphor concentration as the diffusion of a chemical substance. It is also assumed that a portion of the concentration dissolves into the aqueous phase, or is sublimated (see Nagayama et al., 2014, and the references therein). Surface tension gradient dynamics then describe the equation of motion for camphor particles in these studies. In particular, the following system has been studied as a mathematical model for the self-propelled motion of camphor:

$$\begin{cases} u_t = D\Delta u - ku + F(x, x_c(t); r) & x \in \Omega, t > 0, \\ \mu \dot{x}_c = \nabla\gamma(u(t, x_c(t))) & t > 0. \end{cases}$$

Here u denotes the concentration of the camphor layer, Δ denotes the Laplacian, D is the diffusion coefficient, k is the combined rate of sublimation and dissolution of the camphor, r is the radius of the camphor disk, $x_c(t)$ is the center of the camphor disk, μ is a surface viscosity and Ω is the region of the water surface. The source term in the bulk equation is assumed to take the form of a characteristic function:

$$F(x, x_c; r) = \begin{cases} s_0 u_0, & |x - x_c| \le r, \\ 0, & |x - x_c| > r, \end{cases}$$

where s_0 signifies the rate at which camphor is released into the bulk and u_0 expresses a constant source concentration on the surface of the camphor.

Created through experimental observations, the surface tension at the interface between the camphor disk and the water is expressed by the function $\gamma(u)$ (see Nakata et al., 2015):

$$\gamma(u) = \frac{(\gamma_0 - \gamma_1)a^m}{a^m + u^m} + \gamma_1,$$

where γ_0 and γ_1 are surface tension coefficients (representing the surface tension of pure and camphor-saturated water, respectively), a is a positive real number, and m is a positive integer (usually taken as $m = 2$).

These studies also ignore curvature effects when the camphor's motion is restricted to the thin water channel and the following one-dimensional reaction-diffusion system forms the model equation:

$$\begin{cases} u_t = Du_{xx} - ku + F(x, x_c; r), & x \in (0, L),\ t > 0, \\ \mu \dot{x}_c = \frac{\partial}{\partial x}\gamma(u(t, x_c(t))) & t > 0, \end{cases} \tag{1}$$

where L denotes the length of the channel and periodic boundary conditions are imposed:

$$\begin{cases} u(t, 0) = u(t, L) & t > 0 \\ u_x(t, 0) = u_x(t, L) & t > 0. \end{cases}$$

Given $x_c^0 \in [0, L)$ initial conditions are taken as follows:

$$\begin{cases} u(t = 0, x) = 0 & x \in [0, L) \\ x_c(t = 0) = x_c^0. \end{cases}$$

Existence and stability of traveling pulse solutions to the above equation have been studied (Nagayama et al., 2004). The previous study showed the existence of a critical viscosity μ_c at which stationary solutions change their stability. In particular, viscosities larger than μ_c were found to be stable, while traveling pulse solutions with nonzero velocities (one negative and one positive) coexist with the stationary solution at viscosities below this value. The stationary solution is unstable in this region, while the traveling pulses are stable.

4. THE POINT MASS MODEL

This section introduces the model equation used in the current study. The main difference being our treatment of the source terms.

Keeping the mass of the camphor constant as its radius is taken to zero, we form a delta sequence to treat the source term in the bulk equation (1):

$$\delta(x - x_c) = \lim_{r \to 0+} \frac{F(x, x_c; r)}{2 s_0 u_0 r}.$$

The presence of the delta function imparts a jump in the first derivative of the concentration u at x_c so that the equation of motion for x_c becomes:

$$\mu \dot{x}_c = \frac{1}{2}\left\{\gamma_x^+(u(t, x_c(t))) + \gamma_x^-(u(t, x_c(t)))\right\} \qquad t > 0,$$

where the right hand side is the symmetric derivative whose terms express right and left hand derivatives:

$$\gamma_x^+(t,u(x_c(t))) = \lim_{h\to 0+} \frac{\gamma(u(t,x_c(t)+h)) - \gamma(u(t,x_c(t)))}{h}$$

and

$$\gamma_x^-(u(t,x_c(t))) = \lim_{h\to 0-} \frac{\gamma(u(t,x_c(t)+h)) - \gamma(u(t,x_c(t)))}{h}.$$

Our model equation then reads:

$$\begin{cases} u_t = Du_{xx} - ku + c_1\delta(x-x_c), \ x\in[0,L), \ t>0, \\ \mu\dot{x}_c = \frac{1}{2}\{\gamma_x^+(u(t,x_c(t))) + \gamma_x^-(u(t,x_c(t)))\} \ t>0, \end{cases} \tag{2}$$

where c_1 denotes the mass of the camphor. We assume an initial concentration $u(0, x) = 0$ and that the initial location of the camphor is $x_c(0)\in[0,L)$. Periodic boundary conditions are imposed, where the second distributional derivative at the origin is defined to impart the jump in the first derivative whenever $x_c(t)$ is zero (see for example, Estrada & Kanwal, 2002):

$$u_{xx}(t,0) = u_{xx}^-(t,L) + (u_x^+(t,0) - u_x^-(t,L))\delta(x_c) \quad t>0. \tag{3}$$

5. EXISTENCE OF TRAVELING PULSE SOLUTIONS

As has also been remarked in previous studies (Ikura et al., 2013), our experimental observations strongly suggest the existence of traveling pulse solutions within the 1 dimensional water chamber model (2). Such solutions have been confirmed to exist in the characteristic function based model equation (1). Since a main purpose of the current study is to compare the point-mass framework with previous research, we will now investigate this claim. For the sake of simplicity in the analysis, we will take $\gamma(u) = -u$. Our dimensionless model equation then takes the form:

$$\begin{cases} u_t = Du_{xx} - u + c_1\delta(x-x_c), \quad x\in[0,L), \ t>0, \\ \mu\dot{x}_c = -\frac{1}{2}\{u_x^+(x_c(t),t) + u_x^-(x_c(t),t)\} \qquad t>0, \end{cases} \tag{4}$$

where we impose periodic boundary conditions and the following change of variables have been used (we have rewritten the model equations using the same variables, for the sake of clarity):

$$\tau = kt, \quad X = \sqrt{\frac{k}{D}}x, \quad X_c = \sqrt{\frac{k}{D}}x_c, \quad \mu = \frac{1}{\sqrt{kD}}\eta.$$

In particular, we have the following result:

$$\tau = kt, \quad X = \sqrt{\frac{k}{D}}x, \quad X_c = \sqrt{\frac{k}{D}}x_c, \quad \mu = \frac{1}{\sqrt{kD}}\eta.$$

Proposition. *Without loss of generality, let $L = 1$, the initial condition $x_c^0 = 0$ and v be a nonzero velocity. If μ satisfies*

$$\mu < \frac{c_1}{4}\left(1 - \frac{2}{(e-1)^2}\right),$$

then there exist traveling pulse solutions with velocities $\pm v$.

Proof. Let x_c move with velocity v in the x direction and introduce a moving frame of reference via the change of variables:

$$z = x - vt.$$

Also, define U to be the following function:

$$U(z) := u(t, x - vt).$$

Upon computation, one has:

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial z}\frac{\partial z}{\partial t} = -v\frac{\partial U}{\partial z} = -v\frac{dU}{dz}$$
$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial z}\frac{\partial z}{\partial x} = \frac{dU}{dz}$$
$$\frac{\partial^2 U}{\partial x^2} = \frac{d^2 U}{dz^2}.$$

Since x_c is a function of t which starts at time $t = 0$ from x_c^0, if x_c moves with a constant velocity v, then one has

$$\delta(x-x_c) = \delta(x-x_c(t)) = \delta(x-(x_c^0+vt)). \tag{5}$$

Under our change of variables, the delta function term is thus expressed:

$$\delta(x-x_c) = \delta(z+vt-(x_c^0+vt)) = \delta(z-x_c^0).$$

Collecting these facts yields the following partial differential equation from the bulk equation:

$$\frac{d^2U}{dz^2} + v\frac{dU}{dz} - U + c_1\delta(z-x_c^0) = 0.$$

We compute:

$$\frac{dx_c}{dt} = \frac{d(x_c^0+vt)}{dt} = \frac{d(vt)}{dt} = v, \tag{6}$$

and hence the above considerations yield the following equation from the ordinary differential equation:

$$\mu v = -\frac{1}{2}\left\{U_z^+(0) + U_z^-(0)\right\}.$$

Consequently, we have the following problem for U under periodic boundary conditions:

$$\begin{cases} U'' + vU' - U + c_1\delta(z - x_c^0) = 0 & z \in [0,1) \\ \mu v = -\frac{1}{2}\left\{U_z^+(0) + U_z^-(0)\right\}. \end{cases} \quad (7)$$

The presence of the delta function imparts a jump in the first derivative of U so that the second distributional derivative is expressed:

$$U'' = \ddot{U} + [\dot{U}(0+) - \dot{U}(1-)]\delta(0), \quad (8)$$

where $\dot{U}$ denotes the ordinary derivative. The general solution of the above ordinary differentia equation within (0, 1) is given by

$$U(z) = C_+ e^{r^+ z} + C_- e^{r^- z} \quad (9)$$

where $r^{\pm}$ denote roots of the characteristic equation

$$r^{\pm} = \frac{-v \pm \sqrt{v^2+4}}{2}. \quad (10)$$

We thus choose the coefficients to impose continuity in the solution at $z = 0$ and $z = 1$, and such that the appropriate jump in the first derivative appears:

$$\begin{pmatrix} 1 - e^{r^+ L} & 1 - e^{r^- L} \\ r^+ - r^+ e^{r^+ L} & r^- - r^- L \end{pmatrix} \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} 0 \\ -c_1 \end{pmatrix}. (11)$$

The coefficients are then found:

$$C_+ = -\frac{c_1}{\sqrt{v^2+4}\left(e^{-\frac{1}{2}L\left(\sqrt{v^2+4}+v\right)} - 1\right)},$$
$$C_- = \frac{c_1}{\sqrt{v^2+4}\left(e^{\frac{1}{2}L\left(\sqrt{v^2+4}-v\right)} - 1\right)}, \quad (12)$$

and the solution of the ordinary differential equation is expressed:

$$U(z) = -\frac{c_1 e^{-\frac{1}{2}\left(\sqrt{v^2+4}+v\right)z}}{\left(e^{\sqrt{v^2+4}} - 1\right)\sqrt{v^2+4}\left(\cosh\left(\frac{v}{2}\right) - \cosh\left(\frac{\sqrt{v^2+4}}{2}\right)\right)} - \frac{\left(-e^{\sqrt{v^2+4}z} + e^{\frac{1}{2}\left(\sqrt{v^2+4}(2z+1)+v\right)} + e^{\sqrt{v^2+4}} - e^{\frac{1}{2}\left(\sqrt{v^2+4}+v\right)}\right)}{\left(e^{\sqrt{v^2+4}} - 1\right)\sqrt{v^2+4}\left(\cosh\left(\frac{v}{2}\right) - \cosh\left(\frac{\sqrt{v^2+4}}{2}\right)\right)} - \frac{\sinh\left(\frac{\sqrt{v^2+4}}{2}\right)}{\left(e^{\sqrt{v^2+4}} - 1\right)\sqrt{v^2+4}\left(\cosh\left(\frac{v}{2}\right) - \cosh\left(\frac{\sqrt{v^2+4}}{2}\right)\right)}.$$

Computing the symmetric derivative at $x_c^0 = 0$ gives:

$$-\frac{1}{2}\left\{U_z^+(0) + U_z^-(0)\right\} = c_1\left(\frac{v\sinh\left(\frac{\sqrt{v^2+4}}{2}\right)}{\sqrt{v^2+4}\left(2\cosh\left(\frac{v}{2}\right) - 2\cosh\left(\frac{\sqrt{v^2+4}}{2}\right)\right)} - \frac{\sinh\left(\frac{v}{2}\right)}{2\cosh\left(\frac{v}{2}\right) - 2\cosh\left(\frac{\sqrt{v^2+4}}{2}\right)}\right),$$

and so by (7), we have

$$\mu v = c_1\left(\frac{v\sinh\left(\frac{\sqrt{v^2+4}}{2}\right)}{\sqrt{v^2+4}\left(2\cosh\left(\frac{v}{2}\right) - 2\cosh\left(\frac{\sqrt{v^2+4}}{2}\right)\right)} - \frac{\sinh\left(\frac{v}{2}\right)}{2\cosh\left(\frac{v}{2}\right) - 2\cosh\left(\frac{\sqrt{v^2+4}}{2}\right)}\right).$$

For the above equality to hold when v is nonzero, we find that μ must satisfy

$$\mu < \frac{c_1}{4}\left(1 - \frac{2}{(e-1)^2}\right).$$

We also observe that the traveling pulse solutions are unique (up to a sign change) and remark the existence of stationary solutions (corresponding to $v = 0$)

6. STABILITY OF TRAVELING PULSE SOLUTIONS

Having shown the existence of traveling pulse solutions, we now perform their linear stability analysis. In particular, we construct a system of differential equations for investigating their stability. The camphor moves with a velocity v and we again utilize a moving frame of reference.

If we assume the solution of (4) is a linear combination of functions:

$$\begin{cases} u(t,x) = U(x - vt) + \varepsilon g(t, x - vt) \\ x_c(t) = x_c^0 + vt + \varepsilon\phi(t), \end{cases}$$

where U denotes the solution to problem (7) from the previous section and ε is a small parameter, then a description of the function g can be obtained, as follows. In particular, upon making the change of variables $z = x - vt$, the left hand side of the bulk equation becomes

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial z}\frac{\partial z}{\partial t} + \varepsilon\frac{\partial g}{\partial t} + \varepsilon\frac{\partial g}{\partial z}\frac{\partial z}{\partial t},$$
$$= -v\frac{\partial U}{\partial z} + \varepsilon\frac{\partial g}{\partial t} - \varepsilon v\frac{\partial g}{\partial z}.$$

Similarly, the right hand side is expressed:

$$\frac{\partial^2 u}{\partial x^2} - u + c_1\delta(x - x_c) = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial z}\frac{\partial z}{\partial x}\right)$$
$$+ \varepsilon\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial z}\frac{\partial z}{\partial x}\right) - U(z) - \varepsilon g(t,z)$$
$$+ c_1\delta(x - (x_c^0 + vt + \varepsilon\phi(t))),$$
$$= \frac{\partial^2 U}{\partial z^2}\frac{\partial z}{\partial x} + \varepsilon\frac{\partial^2 g}{\partial z^2}\frac{\partial z}{\partial x} - U(z) - \varepsilon g(t,z)$$
$$+ c_1\delta(z - (x_c^0 + \varepsilon\phi(t))),$$
$$= \frac{\partial^2 U}{\partial z^2} + \varepsilon\frac{\partial^2 g}{\partial z^2} - U(z) - \varepsilon g(t,z)$$
$$+ c_1\delta(z - (x_c^0 + \varepsilon\phi(t))).$$

The bulk equation then becomes:

$$-v\frac{\partial U}{\partial z} + \varepsilon\frac{\partial g}{\partial t} - \varepsilon v\frac{\partial g}{\partial z} = \frac{\partial^2 U}{\partial z^2} + \varepsilon\frac{\partial^2 g}{\partial z^2}$$
$$- U(z) - \varepsilon g(t,z) + c_1\delta(z - (x_c^0 + \varepsilon\phi(t))).$$

Without loss of generality we can assume $x_c^0 = 0$. The resulting equation becomes:

$$-v\frac{\partial U}{\partial z} + \varepsilon\frac{\partial g}{\partial t} - \varepsilon v\frac{\partial g}{\partial z} = \frac{\partial^2 U}{\partial z^2} + \varepsilon\frac{\partial^2 g}{\partial z^2}$$
$$- U(z) - \varepsilon g(t,z) + c_1\delta(z - \varepsilon\phi(t)),$$

which can be written:

$$\frac{\partial^2 U}{\partial z^2} + v\frac{\partial U}{\partial z} - U(z) + c_1\delta(z - \varepsilon\phi(t))$$
$$= \varepsilon\left(\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial z^2} - v\frac{\partial g}{\partial z} + g(t,z)\right). \quad (13)$$

Since the traveling pulse solution satisfies:

$$\frac{\partial^2 U}{\partial z^2} + v\frac{\partial U}{\partial z} - U(z) = -c_1\delta(z),$$

this can be used in (13) to yield

$$c_1\left(\delta(z - \varepsilon\phi(t)) - \delta(z)\right)$$
$$= \varepsilon\left(\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial z^2} - v\frac{\partial g}{\partial z} + g(t,z)\right).$$

The properties of distributional derivatives then yield:

$$-c_1\phi(t)\delta'(z) = \frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial z^2} - v\frac{\partial g}{\partial z} + g(t,z) + \mathcal{O}(\varepsilon), \quad (14)$$

where $\delta(z)$ denotes the distributional derivative of the delta function. Taking $\varepsilon \to 0$ gives

$$-c_1\phi(t)\delta'(z) = \frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial z^2} - v\frac{\partial g}{\partial z} + g(t,z). \quad (15)$$

Next we will consider the ordinary differential equation within (4). We compute

$$\mu\frac{d}{dt}(x_c^0 + vt + \varepsilon\phi(t))$$
$$= -\frac{1}{2}\left\{u_x^+(x_c(t),t) + u_x^-(x_c(t),t)\right\}$$
$$= -\lim_{h\to 0+}\frac{u(t,x_c+h) - u(t,x_c-h)}{2h}$$
$$= -\lim_{h\to 0+}\left(\frac{1}{2h}\left(U(x_c - vt + h) + \varepsilon g(t, x_c - vt + h)\right.\right.$$
$$\left.\left.-U(x_c - vt - h) - \varepsilon g(t, x_c - vt - h)\right)\right).$$

Since $x_c(t) = x_c^0 + vt + \varepsilon\phi(t)$ and $x_c^0 = 0$, we have

$$\mu v + \varepsilon\frac{d\phi}{dt}$$
$$= -\lim_{h\to 0+}\left(\frac{1}{2h}(U(\varepsilon\phi(t) + h) - U(\varepsilon\phi(t) - h)\right.$$
$$\left.+ \varepsilon g(t, \varepsilon\phi(t) + h) - \varepsilon g(t, \varepsilon\phi(t) - h))\right). \quad (16)$$

Formally expanding U (under periodic boundary conditions) in the above gives

$$U(\varepsilon\phi(t) + h) - U(\varepsilon\phi(t) - h)$$
$$= \left(U(h) + \varepsilon\phi(t)\frac{dU(h)}{dz}\right)$$
$$- \left(U(-h) + \varepsilon\phi(t)\frac{dU(-h)}{dz}\right) + \mathcal{O}(\varepsilon^2)$$
$$= (U(h) - U(-h)) + \varepsilon\phi(t)\left(\frac{dU(h)}{dz} - \frac{dU(-h)}{dz}\right)$$
$$+ \mathcal{O}(\varepsilon^2) = (U(h) - U(0) + U(0) - U(-h))$$
$$+ \varepsilon\phi(t)\left(\frac{dU(h)}{dz} - \frac{dU(0)}{dz} + \frac{dU(0)}{dz} - \frac{dU(-h)}{dz}\right)$$
$$+ \mathcal{O}(\varepsilon^2).$$

Dividing the last expression by $2h$ and taking the limit gives

$$\lim_{h\to 0+}\frac{1}{2h}\left(U(\varepsilon\phi(t) + h) - U(\varepsilon\phi(t) - h)\right)$$
$$= \frac{1}{2}(U'_+(0) + U'_-(0)) + \varepsilon\phi(t)\frac{1}{2}(U''_+(0) + U''_-(0))$$
$$+ \mathcal{O}(\varepsilon^2).$$

Similarly, expanding g (also under periodic boundary conditions) gives

$$g(t,\varepsilon\phi(t)+h)-g(t,\varepsilon\phi(t)-h)$$
$$=\left(g(t,h)+\varepsilon\phi(t)\frac{dg(t,h)}{dz}\right)$$
$$-\left(g(t,-h)+\varepsilon\phi(t)\frac{dg(t,-h)}{dz}\right)+\mathcal{O}(\varepsilon^2)$$
$$=(g(t,h)-g(t,-h))+\varepsilon\phi(t)\left(\frac{dg(t,h)}{dz}-\frac{dg(t,-h)}{dz}\right)$$
$$+\mathcal{O}(\varepsilon^2),$$
$$=(g(t,h)-g(t,0)+g(t,0)-g(t,-h))$$
$$+\varepsilon\phi(t)\left(\frac{dg(t,h)}{dz}-\frac{dg(t,0)}{dz}+\frac{dg(t,0)}{dz}-\frac{dg(t,-h)}{dz}\right)$$
$$+\mathcal{O}(\varepsilon^2).$$

Again dividing the last expression by $2h$ and taking the limit gives

$$\lim_{h\to0+}\frac{1}{2h}\left(g(t,\varepsilon\phi(t)+h)-g(t,\varepsilon\phi(t)-h)\right)$$
$$=\frac{1}{2}(g'_+(t,0)+g'_-(t,0))+\varepsilon\phi(t)\frac{1}{2}(g''_+(t,0)$$
$$+g''_-(t,0))+\mathcal{O}(\varepsilon^2).$$

Hence equation (16) becomes

$$\mu v+\varepsilon\frac{d\phi}{dt}=-\frac{1}{2}(U'_+(0)+U'_-(0))-\varepsilon\phi(t)\frac{1}{2}(U''_+(0)$$
$$+U''_-(0))-\frac{\varepsilon}{2}(g'_+(t,0)+g'_-(t,0))+\mathcal{O}(\varepsilon^2).$$

Since $U(z)$ is a traveling pulse, one has

$$\mu v=-\frac{1}{2}\Big(U'_+(0)+U'_-(0)\Big),$$

and it follows that

$$\frac{d\phi}{dt}=-\frac{1}{2}\Big(U''_+(0)+U''_-(0)\Big)\phi(t)-\frac{1}{2}\Big(g'_+(t,0)$$
$$+g'_-(t,0)\Big)+\mathcal{O}(\varepsilon). \tag{17}$$

Taking the limit $\varepsilon\to0$ in Equations (14) and (17) expresses the following system (under periodic boundary conditions):

$$\begin{cases}\dfrac{\partial g}{\partial t}=\dfrac{\partial^2 g}{\partial z^2}+v\dfrac{\partial g}{\partial z}-g(t,z)-c_1\phi(t)\delta'(z), \\ \qquad\qquad z\in[0,L),t>0\\ \dfrac{d\phi}{dt}=-\dfrac{1}{2}\Big(U''_+(0)+U''_-(0)\Big)\phi(t)\\ \quad-\dfrac{1}{2}\Big(g'_+(t,0)+g'_-(t,0)\Big)\qquad t>0.\end{cases} \tag{18}$$

6.1. Stability analysis via an eigenvalue problem

In this section we will construct the eigenvalue problem for analyzing the linear stability of the traveling pulses. In particular, we assume modes of the following form:

$$g(t,x)=\hat{g}(z)e^{\lambda t}$$
$$\phi(t)=\hat{\phi}e^{\lambda t},$$

where $\hat{g}$ is an $L-$ periodic function, and $\hat{\phi}$ is a scalar. Substituting these into (18) gives the following system for $\lambda\in\mathbf{C}$:

$$\begin{cases}\lambda e^{\lambda t}\hat{g}=e^{\lambda t}\left(\hat{g}''+v\hat{g}'-\hat{g}\right)-c_1\hat{\phi}e^{\lambda t}\delta'(z),\\ \lambda\mu\hat{\phi}e^{\lambda t}=-\tilde{C}\hat{\phi}e^{\lambda t}-\dfrac{1}{2}(\hat{g}'_+(0)+\hat{g}'_-(0))e^{\lambda t},\end{cases} \tag{19}$$

where we have denoted $\frac{1}{2}\Big(U''_+(0)+U''_-(0)\Big)$ by the coefficient $\tilde{C}$ and whose exact value is

$$\tilde{C}=\frac{1}{2}\left(U''_+(0)+U''_-(0)\right)$$
$$=\frac{v\sqrt{v^2+4}\sinh\left(\frac{Lv}{2}\right)-(v^2+2)\sinh\left(\frac{1}{2}L\sqrt{v^2+4}\right)}{2\sqrt{v^2+4}\left(\cosh\left(\frac{Lv}{2}\right)-\cosh\left(\frac{1}{2}L\sqrt{v^2+4}\right)\right)}.$$

Upon dividing both sides of (19) by $e^{\lambda t}$, we obtain the following eigenvalue problem:

$$\begin{cases}\lambda\hat{g}=\hat{g}''+v\hat{g}'-\hat{g}-c_1\hat{\phi}\delta'(z),\\ \lambda\mu\hat{\phi}=-\tilde{C}\hat{\phi}-\dfrac{1}{2}(\hat{g}'_+(0)+\hat{g}'_-(0)),\end{cases} \tag{20}$$

which is equivalent to

$$\begin{cases}\hat{g}''+v\hat{g}'-(1+\lambda)\hat{g}-c_1\hat{\phi}\delta'(z)=0,\\ \left(\lambda\mu+\tilde{C}\right)\hat{\phi}=-\dfrac{1}{2}\left(\hat{g}'_+(0)+\hat{g}'_-(0)\right).\end{cases} \tag{21}$$

Outside the support of the delta function, the general solution of (21) is

$$\hat{g}(z)=C_1e^{\frac{1}{2}z\left(-\sqrt{4\lambda+v^2+4}-v\right)}+C_2e^{\frac{1}{2}z\left(\sqrt{4\lambda+v^2+4}-v\right)}, \tag{22}$$

and we remark that the first and second distributional derivatives of $\hat{g}$ are given by the formulas:

$$\hat{g}'=\dot{\hat{g}}+\{\hat{g}(0+0)-\hat{g}(0-0)\}\delta(z)$$
$$\hat{g}''=\ddot{\hat{g}}+\{\dot{\hat{g}}(0+0)-\dot{\hat{g}}(0-0)\}\delta(z)$$
$$+\{\hat{g}(0+0)-\hat{g}(0-0)\}\delta'(z),$$

where $\dot{\hat{g}}$ and $\ddot{\hat{g}}$ denote standard differentiations.

We therefore choose the coefficients in (22) to impart the appropriate jumps for equation (20):

$$\begin{cases} (\hat{g}'(0+0) - \hat{g}'(L-0)) + v(\hat{g}(0+0) - \hat{g}(L-0)) \\ \quad = 0, \\ \hat{g}(0+0) - \hat{g}(L-0) = -c_1\hat{\phi}. \end{cases} \tag{23}$$

By the first equation above, $\hat{g}(z)$ is given by

$$\hat{g}(z) = \frac{C_1}{\left(\sqrt{4\lambda+v^2+4}+v\right)\left(e^{\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}-v\right)}-1\right)} \left(e^{-\frac{1}{2}(L+z)\left(\sqrt{4\lambda+v^2+4}+v\right)}\left(\left(v-\sqrt{4\lambda+v^2+4}\right) \times e^{\left(\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}+v\right)+z\sqrt{4\lambda+v^2+4}\right)} + \left(\sqrt{4\lambda+v^2+4}+v\right)e^{L\sqrt{4\lambda+v^2+4}} - \left(\sqrt{4\lambda+v^2+4}+v\right)e^{\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}+v\right)}\right)\right).$$

Using this, the left hand side of the second equation in (23) becomes

$$\hat{g}(0+0) - \hat{g}(L-0) = \frac{2C_1\sqrt{4\lambda+v^2+4}\,e^{-\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}+v\right)}}{\sqrt{4\lambda+v^2+4}+v} \frac{\left(e^{\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}+v\right)}-1\right)}{\sqrt{4\lambda+v^2+4}+v}. \tag{24}$$

Moreover, by the second equation in (21), the right hand side of the second equation in (23) becomes

$$-c_1\hat{\phi} = c_1\frac{1}{2\left(\lambda\mu+\tilde{C}\right)}\left(\hat{g}'_+(0)+\hat{g}'_-(0)\right). \tag{25}$$

Upon computation, we have

$$\hat{g}'_+(0)+\hat{g}'_-(0) = \frac{4C_1\left((2\lambda+v^2+2)\sinh\left(\frac{1}{2}L\sqrt{4\lambda+v^2+4}\right) - v\sqrt{4\lambda+v^2+4}\sinh\left(\frac{Lv}{2}\right)\right)}{\left(\sqrt{4\lambda+v^2+4}+v\right)\left(e^{\frac{Lv}{2}} - e^{\frac{1}{2}L\sqrt{4\lambda+v^2+4}}\right)}$$

By the above, the right hand side of (25) satisfies

$$c_1\frac{1}{2\left(\lambda\mu+\tilde{C}\right)}\left(\hat{g}'_+(0)+\hat{g}'_-(0)\right) = \frac{N}{D}$$

where

$$N = -c_1C_1\sqrt{v^2+4}\left(\sqrt{4\lambda+v^2+4}+v\right)e^{-\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}+v\right)}\left(e^{L\sqrt{4\lambda+v^2+4}}-1\right) \times\left(\cosh\left(\frac{Lv}{2}\right) - \cosh\left(\frac{1}{2}L\sqrt{v^2+4}\right)\right)$$

$$D = \left(e^{\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}-v\right)}-1\right)\left(2\lambda\mu\sqrt{v^2+4}\cosh\left(\frac{Lv}{2}\right) - 2\lambda\mu\sqrt{v^2+4}\cosh\left(\frac{1}{2}L\sqrt{v^2+4}\right) + v\sqrt{v^2+4}\sinh\left(\frac{Lv}{2}\right) - (v^2+2)\sinh\left(\frac{1}{2}L\sqrt{v^2+4}\right)\right).$$

Using (24), we obtain

$$\frac{2C_1\sqrt{4\lambda+v^2+4}\,e^{-\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}+v\right)}\left(e^{\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}+v\right)}-1\right)}{\sqrt{4\lambda+v^2+4}+v} = \frac{N}{D}.$$

Here we note that taking $C_1 = 0$ corresponds to the trivial solution $\hat{g}(z) = 0$, so we take $C_1 = 0$. Upon dividing the previous equation by C_1, we have

$$\frac{2\sqrt{4\lambda+v^2+4}\,e^{-\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}+v\right)}\left(e^{\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}+v\right)}-1\right)}{\sqrt{4\lambda+v^2+4}+v} = \frac{N'}{D}$$

where

$$N' = -c_1\sqrt{v^2+4}\left(\sqrt{4\lambda+v^2+4}+v\right)e^{-\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}+v\right)}\left(e^{L\sqrt{4\lambda+v^2+4}}-1\right) \times\left(\cosh\left(\frac{Lv}{2}\right) - \cosh\left(\frac{1}{2}L\sqrt{v^2+4}\right)\right)$$

$$D = \left(e^{\frac{1}{2}L\left(\sqrt{4\lambda+v^2+4}-v\right)}-1\right)\left(2\lambda\mu\sqrt{v^2+4}\cosh\left(\frac{Lv}{2}\right) - 2\lambda\mu\sqrt{v^2+4}\cosh\left(\frac{1}{2}L\sqrt{v^2+4}\right) + v\sqrt{v^2+4}\sinh\left(\frac{Lv}{2}\right) - (v^2+2)\sinh\left(\frac{1}{2}L\sqrt{v^2+4}\right)\right).$$

Upon simplification, we find that the above can be expressed as the following nonlinear eigen- value problem for λ:

$$\frac{1}{2}c_1\left(\frac{v\sqrt{4\lambda+v^2+4}\sinh\left(\frac{Lv}{2}\right) - (2\lambda+v^2+2)\sinh\left(\frac{1}{2}L\sqrt{4\lambda+v^2+4}\right)}{\sqrt{4\lambda+v^2+4}\left(\cosh\left(\frac{Lv}{2}\right) - \cosh\left(\frac{1}{2}L\sqrt{4\lambda+v^2+4}\right)\right)} + \frac{(v^2+2)\sinh\left(\frac{1}{2}L\sqrt{v^2+4}\right) - v\sqrt{v^2+4}\sinh\left(\frac{Lv}{2}\right)}{\sqrt{v^2+4}\left(\cosh\left(\frac{Lv}{2}\right) - \cosh\left(\frac{1}{2}L\sqrt{v^2+4}\right)\right)}\right) = \lambda\mu.$$

The stability of the traveling pulse solutions is thus contained in the problem of finding λ to satisfying $f(\mu, v, \lambda) = 0$, where

$$f(\mu, v, \lambda) := \frac{1}{2}c_1\left(\frac{v\sqrt{4\lambda+v^2+4}\sinh\left(\frac{Lv}{2}\right) - (2\lambda+v^2+2)\sinh\left(\frac{1}{2}L\sqrt{4\lambda+v^2+4}\right)}{\sqrt{4\lambda+v^2+4}\left(\cosh\left(\frac{Lv}{2}\right) - \cosh\left(\frac{1}{2}L\sqrt{4\lambda+v^2+4}\right)\right)} + \frac{(v^2+2)\sinh\left(\frac{1}{2}L\sqrt{v^2+4}\right) - v\sqrt{v^2+4}\sinh\left(\frac{Lv}{2}\right)}{\sqrt{v^2+4}\left(\cosh\left(\frac{Lv}{2}\right) - \cosh\left(\frac{1}{2}L\sqrt{v^2+4}\right)\right)}\right) - \lambda\mu. \tag{26}$$

6.2. Stability of the stationary solution, $v = 0$

Here we will assume that $c_1 = 1$, $L = 1$, and $v = 0$ (the profile of the stationary solution is shown in figure 4). Then the function (26) is expressed:

$$f(\mu, 0, \lambda) = \frac{1}{2}\left(-2\lambda\mu + \sqrt{\lambda+1}\coth\left(\frac{\sqrt{\lambda+1}}{2}\right) - \frac{(1+e)}{e-1}\right).$$

Figure 5 illustrates the nullclines of f for the stationary solution, $v = 0$. Here, the existence of the eigenvalue $\lambda = 0$ corresponds to the camphor's translational degree of freedom. The remaining eigenvalues are obtained numerically using Newton's method, where we have plotted the real part of the eigenvalue with maximal real part.

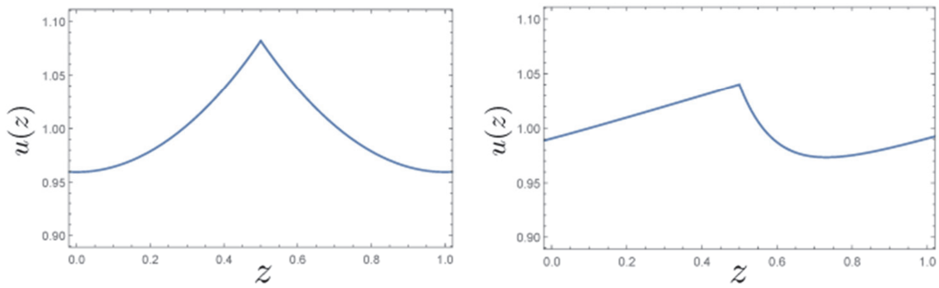

Fig. 4. *(Left) A stationary solution, v = 0. (Right) A traveling pulse solution, v = 10.*

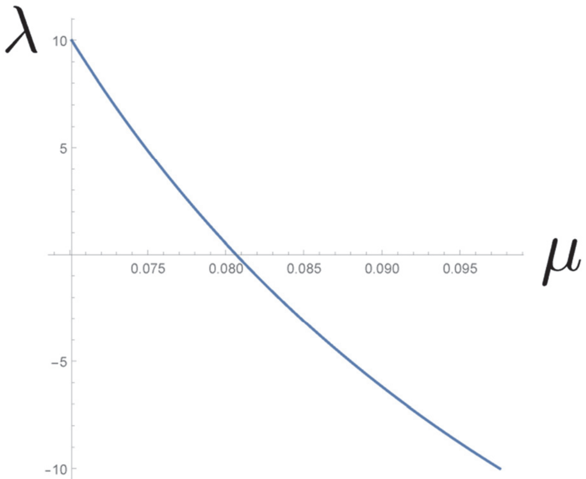

Fig. 5. *Solution structure of f(λ,,0, μ) = 0 (stability of the stationary pulse with respect to μ.*

Research into the model equation (1), which utilized characteristic function source terms, has shown the existence of a critical parameter where λ changes its sign (and hence where traveling pulses change their stability). Our model equation also displays this feature, and the precise value of the critical parameter is found to be:

$$\mu_c = \frac{1}{4}\left(1 - \frac{2}{(e-1)^2}\right) \approx 0.080652.$$

As a result, we obtain the following information about the stability of the stationary solutions:

- The stabilityof the traveling pulse changes when μ takes the value $\mu_c = \frac{1}{4}\left(1 - \frac{2}{(e-1)^2}\right)$.

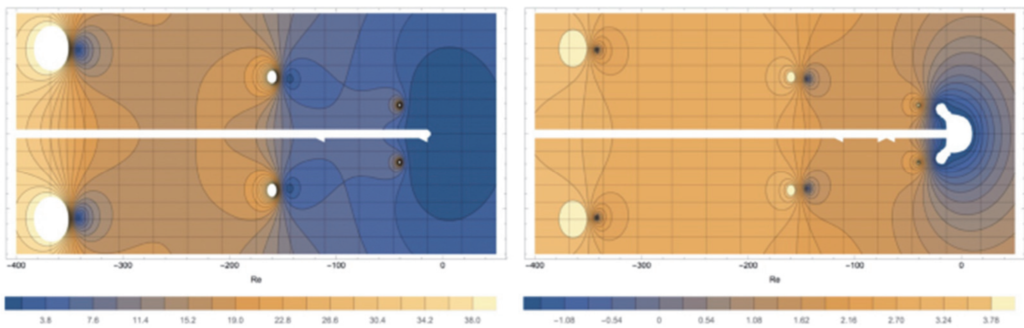

Fig. 6. *(Left) The modulus* $|f(\mu(v), v, \lambda)|$, *when v = 3. (Right) Its logarithm.*

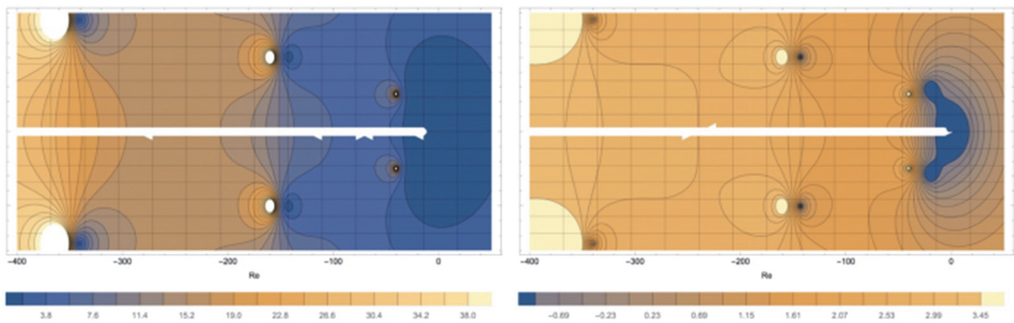

Fig. 7. *(Left) The modulus* $|f(\mu(v), v, \lambda)|$, *when v = 4. (Right) Its logarithm.*

- The traveling pulse is stable whenever μ is greater than μ_c,
- The traveling pulse is unstable whenever μ is less than μ_c.

These facts are also collected into figure 8.

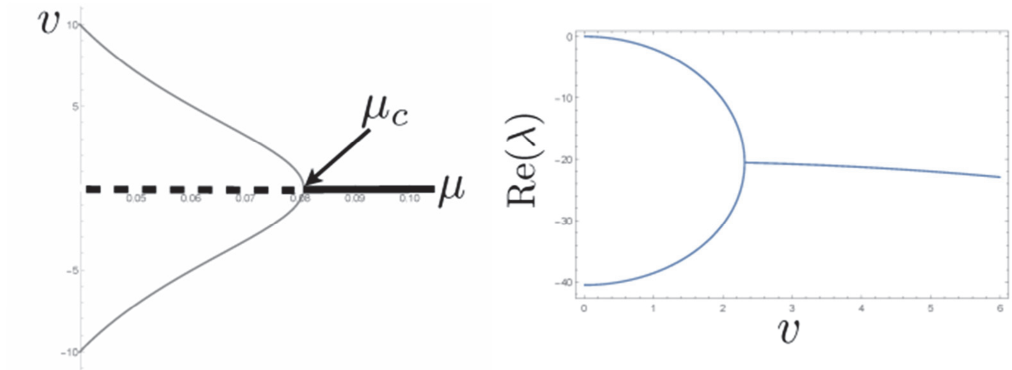

Fig. 8. *(Left) Stability of traveling and standing pulses. The dotted lines express unstable so- lutions, while solid lines indicate stable solutions. (Right) Real part of the eigenvalues obtained via our Newton's method calculations.*

6.3. Stability of traveling pulses with nonzero velocity

We will analyze the stability of traveling pulses with nonzero velocity (see figure 4 for the profile of the concentration when $c_1 = 1$, $L = 1$, and $v = 10$). In particular, we note that v uniquely determines μ so that we can define the function $\mu(v) : \mathbb{R} \to \mathbb{R}$

$$\mu(v) = -\frac{c_1\left(\sqrt{v^2+4}\,e^{\frac{1}{2}L\left(\sqrt{v^2+4}-v\right)}\left(e^{Lv}-1\right)-ve^{L\sqrt{v^2+4}}+v\right)}{2v\sqrt{v^2+4}\left(e^{L\sqrt{v^2+4}}-2e^{\frac{1}{2}L\sqrt{v^2+4}}\cosh\left(\frac{Lv}{2}\right)+1\right)}.$$

Therefore, when v is nonzero we can substitute $\mu(v)$ into $f(\mu, v, \lambda)$ to obtain

$$f(\mu, v, \lambda) = f(\mu(v), v, \lambda) =$$
$$\frac{1}{2}c_1\left(\frac{v\left(\lambda+v^2+2\right)\sinh\left(\frac{1}{2}L\sqrt{v^2+4}\right)-\sqrt{v^2+4}\left(\lambda+v^2\right)\sinh\left(\frac{Lv}{2}\right)}{v\sqrt{v^2+4}\left(\cosh\left(\frac{Lv}{2}\right)-\cosh\left(\frac{1}{2}L\sqrt{v^2+4}\right)\right)}+\frac{v\sqrt{4\lambda+v^2+4}\sinh\left(\frac{Lv}{2}\right)-\left(2\lambda+v^2+2\right)\sinh\left(\frac{1}{2}L\sqrt{4\lambda+v^2+4}\right)}{\sqrt{4\lambda+v^2+4}\left(\cosh\left(\frac{Lv}{2}\right)-\cosh\left(\frac{1}{2}L\sqrt{4\lambda+v^2+4}\right)\right)}\right).$$

Hence we arrive at an eigenvalue problem for λ. satisfying $f(\mu(v), v, \lambda) = 0$.

6.4. Computational analysis of the eigenvalue problem

As before, we take $L = 1$ and $c_1 = 1$. We first investigate the case $v = 3$. Allowing λ.. to take complex values and examining the modulus of f enables us to view the zeros of f in the complex plane (see figure 6). Here we observe that solutions of the eigenvalue problem have on-positive real parts, and we remark that the location of solutions correspond to the dark regions on the left hand side of the complex plane. Moreover, extending the range of values for the real part of λ further into the positive domain reveals no new zeros. This observation also holds when $v = 4$ (see figure 7).

We therefore numerically confirm these observations over a wider range of velocities, $v \in [0, 6]$. In particular, the eigenvalue problem is solved by means of Newton's method (extended to allow for complex valued λ). We observe that, when started from an initial condition with positive real part, Newton's method converges to the zero eigenvalue. We thus prepare two complex conjugate initial conditions with negative real parts that are significantly close to the first maximal eigenvalues. Newton's method then converges to the eigenvalue with maximal real part. We make a small change the velocity v and again use Newton's method with initial conditions starting from the roots obtained in the previous computation. Our computations reveal that the real part of the maximal eigenvalue is non-positive over the range of velocities [0, 6], confirming the stability of traveling pulses with μ less than μ_c (see figure 8).

7. CONCLUSION

Experimental observations led us to employ computational and mathematical techniques for investigating a model equation which describes the self-propelled motion of camphor particles atop water. Our model equation described the location of the camphor particles as point masses and was compared to results from related studies which described the source terms using char- acteristic functions. We showed the existence of stationary and traveling pulse solutions and analyzed their linear stability. Our findings were found to be in agreement with those obtained in previous studies.

We would like to generalize our results to the case of nonlinear surface tension functions, and we aim to make use of the delta function approach to model motions in higher dimensions in the near future. It is also interesting to consider the applicability of related line masses for modeling interfacial motions involving thin strings and filaments.

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ROZWIĄZANIA W POSTACI RUCHOMEGO PULSU DLA MODELU PUNKTOWEJ MASY PODCZAS DYFUZJI CZĄSTEK

Streszczenie

W pracy przedstawiono różniczkowe równania cząstkowe opisujące ruch własny cząstek kamfory na powierzchni wody. Równania modelu zaprezentowano w formie systemu reakcji-dyfuzji, w którym człon źródła jest wyrażony jako funkcje delta. Otrzymany system jest modelem punktowej masy dyfuzji cząstek, w którym rolą funkcji delta jest określenie położenia źródeł kamfory. W opracowanym modelu punktowe źródła działają wzajemnie na siebie i poruszają się zgodnie z gradientem pola stężenia. W publikacji omówiono analityczne własności opracowanych równań. Szczególny nacisk położono na badania własności rozwiązania w postaci ruchomego pulsu, którego istnienie ograniczono do rozwiązania zwyczajnych równań różniczkowych sprzężonych z problemem brzegowym. Wykazano istnienie i stabilność rozwiązania. Wyniki badań zostaną porównane z analogicznymi rozwiązaniami wykorzystującymi funkcje charakterystyczne dla członów źródłowych.

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