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HOMOGENIZATION OF ONE DIMENSIONAL ELLIPTIC SYSTEM

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Abstract

The work presents problem of homogenization of one-dimenional mediums of periodic micrsotructure, and more precisely homogenization of one-dimensional elliptic system of layered periodic micrsotructure. The example of suitable parametrized periodic micrstructure, where parameters of homogenized system can be determine, is given. Next, the following inverse problem is considered: in what proportions components should be mix, to obtain homogenized material of given properties.

Key words: homogenization, elliptic system, layered materials, inverse problem

1. INTRODUCTION

Problems concerning homogenization are important from the practical point of view and were analysed in many publications like for example for example (Cioranescu & Donato, 1999; Bielski et al. 2003; Mityushev, 2001; Berlyand & Mityushev, 2005; see also Mitkowski, 2008). The essence of homogenization consists in suitable "washing away" of heterogeneities, what is described with convergence to zero of small parameter $\varepsilon = l / L > 0$, where l is a characteristic dimension of a microstructure and L is a macroscopic characteristic dimension. Limits of sequences do not depend on choice of ε_n. In this paper we will assume $l = L / 2^n$, $n = 0,1,2,3,\ldots$, that is $\varepsilon_n = 1/2^n$. A model homogenization process is shown in figure 1.

By heterogenity we understand for example fibres set in material, microgaps, mixture of two-component materials, etc. Distribution of heterogenities can be random or deterministic so various approaches to homogenization are applied.

In this paper homogenization of one-dimensional mediums (spatial variable $N = 1$) with periodic microstructure is considered. In the general case there is not possibility of precise determination of parameters of homogenized model. There exist only some estimations of these parameters.

In this paper example of elliptic system, where suitable coefficient of conduction can be determined in process of homogenization, is analyzed. The inverse problem is solved effectivelly thanks to analitical formulas, what shows, how to mix the components, to obtain material of given thermal conductivity.

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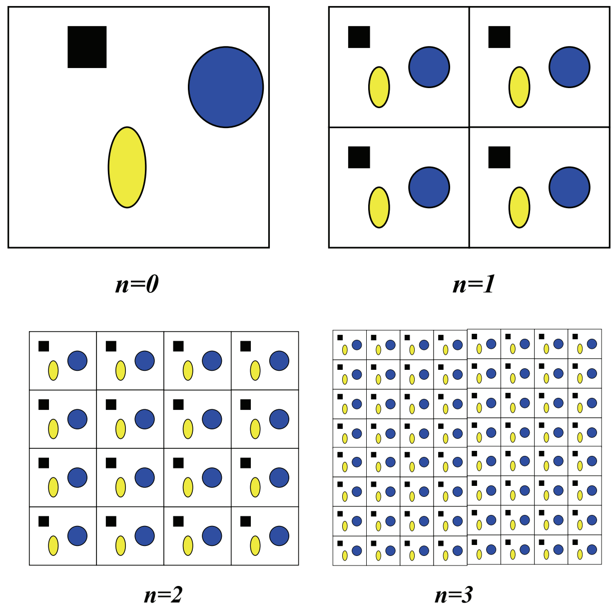

Fig. 1. Homogenization process for n = 0, 1, 2, 3

2. HOMOGENIZATION OF ELLIPTIC SYSTEM

Consider one dimensional ($N = 1$) elliptic boundary problem with parameter $\boldsymbol{\varepsilon} > 0$ given by following equations:

$$\begin{cases} -\dfrac{d}{dx}\left(a^{\varepsilon}(x)\dfrac{du^{\varepsilon}}{dx}\right) = f, \quad x \in (0, d) \\ u^{\varepsilon}(0) = 0 \quad and \quad u^{\varepsilon}(d) = 0 \end{cases} \qquad (1)$$

where $x \in R^N$ for $N = 1$, f is given and $a^{\varepsilon}(x)$ is suitable thermal conductivity of two-component material characterized by factors α and β. First two elements of sequence $a^{\varepsilon}(x)$ for $\varepsilon = \varepsilon_n = 1/2^n$, $n = 0, 1, 2, \ldots$ are shown in figure 2. For $n = 0$ function $a^1(x)$ on $(0, d)$ has the following form:

$$a^1(x) = \begin{cases} \alpha > 0 & for \quad x \in (0, c) \\ \beta > 0 & for\ the\ rest\ x \in (c, d) \end{cases}. \qquad (2)$$

The function $a^1(x)$ is called "basic cell" of homogenization process. The element a^{ε} is given by

$$a^{\varepsilon}(x) = a^1(x/\varepsilon), \quad x \in (0, d) \qquad (3)$$

and a^{ε} is periodic function of period $T = \varepsilon\, d$.

Let u^{ε} be solution of boundary value problem (1) in the microscopic scale x/ε. Now following natural questions appear:

- Does the u^{ε} converge to some limit function $\tilde{u}$?
- If $\tilde{u}$ exists, does it solve some limit boundary value problem in the macroscopic scale of variable x ?
- What is the parameter $\tilde{a}(x)$ of the limit boundary value problem in the macroscopic scale ?
- Is the coefficient $\tilde{a}(x)$ constant ?

The answers for the following questions are given below.

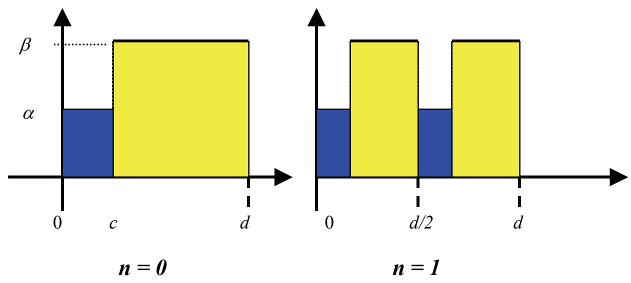

Fig. 2. Two elements of sequence $a^{\boldsymbol{\varepsilon}}$ with $\boldsymbol{\varepsilon} = 1/2^n$, for n = 0 and n = 1

Let $\varepsilon = 1/2^n$ and let $n = 0, 1, 2, \ldots$, then $\varepsilon \to 0$. It is possible to show (Cioranescu & Donato, 1999, p. 33) that for $\varepsilon \to 0$,

$$a^{\varepsilon} \to a = \frac{1}{d}(c\alpha + (d - c)\beta) = \frac{1}{d}\int_0^d a^1(x)dx \qquad (4)$$

weakly in $L^2(0, d)$ and

$$\| a^{\varepsilon} \|^2_{L^2(0,d)} \to \| a \|^2_{L^2(,0,d)} = d[(c\alpha + (d - c)\beta)/d]^2. \qquad (5)$$

Notice, that a in (4) is the mean value of the function a^1. The sequence a^{ε} in $L^2(0, d)$ (generally in Banach space E) converge weakly to a if and only if

$$\int_0^d a^{\varepsilon}(x)v(x)dx \to \int_0^d a(x)v(x)dx \qquad (6)$$

for any integrable function v.

Let $0 < \alpha \le a^1(x) \le \beta < +\infty$. The following theorem about homogenization of elliptic system (1) can be proved (Cioranescu & Donato, 1999, p.96):

Theorem. Let $f \in L^2(0, d)$ and let a^{ε} be determined by (2) and (3). Let $u^{\varepsilon} \in H^1_0(0, d)$ be the solution of the problem (1). Then for $\varepsilon \to 0$,

$$u^{\varepsilon} \to \tilde{u} \text{ weakly in } H_0^1(0,d), \qquad (7)$$

where $\tilde{u}$ is the unique solution of the elliptic boundary Dirichlet problem in $H_0^1(0,d)$ in the form

$$\begin{cases} -\dfrac{d}{dx}\left(\tilde{a}(x)\dfrac{d\tilde{u}}{dx}\right) = f, \quad x \in (0,d) \\ \tilde{u}(0) = 0 \quad and \quad \tilde{u}(d) = 0 \end{cases} \qquad (8)$$

where

$H_0^1(0,d) = \{u : u \in H^1(0,d), u(0) = 0, u(d) = 0\}$,

$H^1(0,d) = \{u : u, du/dx \in L^2(0,d)\}$ are Sobolev spaces and

$$\tilde{a}(x) = \left[\frac{1}{d}\int_0^d \frac{1}{a^1(x)}dx\right]^{-1}. \blacksquare \qquad (9)$$

From (9) evident is, that $\tilde{a}(x) = const.$, so instead $\tilde{a}(x) = const.$, let us write $\tilde{a}$. Model (1) describes heterogeneous system in microscopic scale. Homogenization is characterized by parameter $\varepsilon > 0$. When $\varepsilon \to 0$ mathematically it is homogenization process. Generally (weak convergence in L^2) in the problem (1)

$$\lim_{\varepsilon \to 0}\left(a^{\varepsilon}\frac{du^{\varepsilon}}{dx}\right) \neq \left(\lim_{\varepsilon \to 0} a^{\varepsilon}\right)\lim_{\varepsilon \to 0}\frac{du^{\varepsilon}}{dx} \qquad (10)$$

and then $\tilde{a} \neq a$, where a is given by (4).

The limit $\tilde{u}$ exists and is the unique solution of equation (8) with parameter $\tilde{a}$ and

$$\frac{1}{a^{\varepsilon}} \to \frac{1}{\tilde{a}} = \frac{1}{d}\int_0^d \frac{1}{a^1(x)}dx \quad \text{weakly in} \quad L^{\infty}(0,d). \qquad (11)$$

Equation (8) describes homogenized system, what mathematically is obtained when $\varepsilon \to 0$. Equation (8) is a homogenous model in a macroscopic scale and an approximation of heterogeneous microscopic phenomenon. Boundary problem (8) is called homogenized problem.

Example 1. Let a^{ε} be like in figure 2. Let $d = 2, c = 2/3$ and let $\alpha = 1/2, \beta = 1$. Then $a = \frac{1}{2}\left(\frac{2}{3}\cdot\frac{1}{2} + \frac{4}{3}\cdot 1\right) = \frac{5}{6}$ (see (4)) and from (9) or (11) $\frac{1}{\tilde{a}} = \frac{1}{2}\left(\frac{2}{3}\cdot 2 + \frac{4}{3}\cdot 1\right) = \frac{4}{3}$.

Consequently $\tilde{a} = 3/4$ and $\tilde{u}$ is the unique solution in $H_0^1(0,d)$ of the homogenized elliptic boundary Dirichlet problem (see (8)) given by following equations

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{3}{4}\dfrac{d\tilde{u}}{dx}\right) = f, \quad x \in (0,d) \\ \tilde{u}(0) = 0 \quad and \quad \tilde{u}(d) = 0 \end{cases} \blacksquare \qquad (12)$$

3. ANALYSIS OF PARAMETER OF HOMOGENIZED SYSTEM

Let us consider function a^1 on $(0,d)$ given by (2). Let β and d be fixed. Let $0 < \alpha < \beta < +\infty$ (see figure 2 for $n=0$ or figure 3) and

$$\alpha = \rho\beta, \quad \rho \in (0,1) \quad and \quad c = \gamma d, \quad \gamma \in (0,1). \qquad (13)$$

Using (9) and (13) following formula for parameter $\tilde{a}$, dependent on $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$, is obtained:

$$\tilde{a} = \frac{d\alpha\beta}{c\beta + (d-c)\alpha} = \frac{\rho\beta}{\gamma + (1-\gamma)\rho} = \tilde{a}(\gamma,\rho). \qquad (14)$$

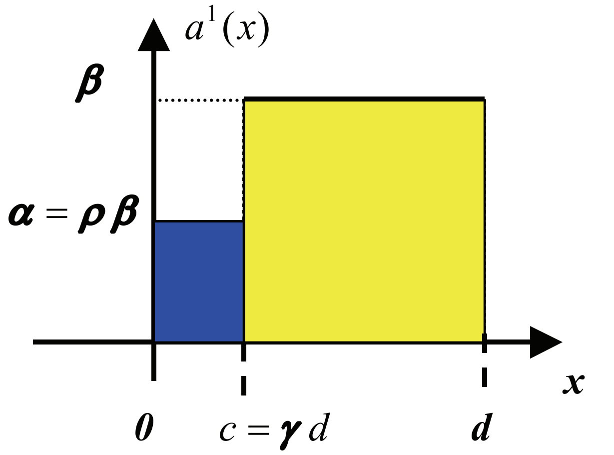

Fig. 3. Basic cell

Let us notice, that fixed value of parameter $\tilde{a}(\gamma,\rho) = \tilde{a} = const.$ of homogenized system (8) can be obtained for various parameters γ and ρ.

4. INVERSE PROBLEM

Let $\beta \in (0,+\infty)$. Now let us consider inverse problem. Let $\tilde{a} \in (0,\beta]$, $\tilde{a} = const.$ We look for γ and ρ such that (see (14))

$$\frac{\rho\beta}{\gamma+(1-\gamma)\rho}=\tilde{a}\ .\ \blacksquare \qquad (15)$$

From (15)

$$\rho=\frac{\tilde{a}\gamma}{\beta-\tilde{a}(1-\gamma)} \quad \text{or} \quad \gamma=\frac{\rho(\beta-\tilde{a})}{\tilde{a}(1-\rho)}. \qquad (16)$$

Let us notice that $\rho(\beta,\tilde{a},\gamma)=\rho(\gamma)$ because β and $\tilde{a}$ are fixed. Similarly $\gamma(\beta,\tilde{a},\rho)=\gamma(\rho)$.

Solution of given inverse problem is pair of parameters: γ and $\rho(\beta,\tilde{a},\gamma)$. Parameter $\gamma\in(0,1)$, then for fixed $\tilde{a}$ there exist inifinite number of pairs γ, $\rho(\gamma)$, such that equation (15) is satysfied. The pair γ and $\rho(\gamma)$ determine the basic cell shown in figure 3.

Example 2. Let $\beta=1$ and let $\tilde{a}=3/4$. Function $\rho(\gamma)$ is shown in figure 4.

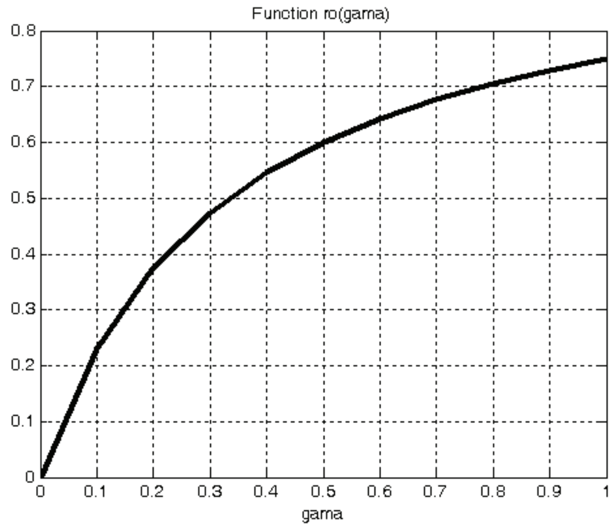

Fig. 4. Function $\boldsymbol{\rho(\gamma)}$ *for* $\tilde{a}=3/4$

For two different values of γ, parameter $\tilde{a}$ is the same. From (16) $\rho(1/3)=1/2$, then $c=d/3$, $\alpha=\beta/2$ and consequently from (15) $\tilde{a}=3/4$ (see Example 1). From (16) $\rho(1/2) = 3/5$ and from (15) $\tilde{a}=3/4$. ■

The inverse problem has an infinite number of solutions. To obtain unique solution suitable optimal problem can be formulated, what is presented in following example.

Example 3. Let L be the circumference of rectangle connected with factor $\boldsymbol{\alpha}$ (see figure 3) and let S be the area of this rectangle, then

$$L=2(\alpha+c)=2(\rho\beta+\gamma\, d) \quad \text{and} \quad S=\alpha\, c=\rho\beta\gamma\, d\ . \qquad (17)$$

If β, L and $\tilde{a}$ are fixed, then

$$S(\gamma)=\rho(\gamma)\beta[L/2-\rho(\gamma)\beta]\ . \qquad (18)$$

It is clear, that the area of rectangle, with fixed circumference, will have maximal value when rectangle will be square, that is, maximum is reached for $\alpha=c$ (see figure 3).

Function $S(\gamma)$ for $d=2, \beta=1, L=1.5$ and $\tilde{a}=3/4$ is shown in figure 5.

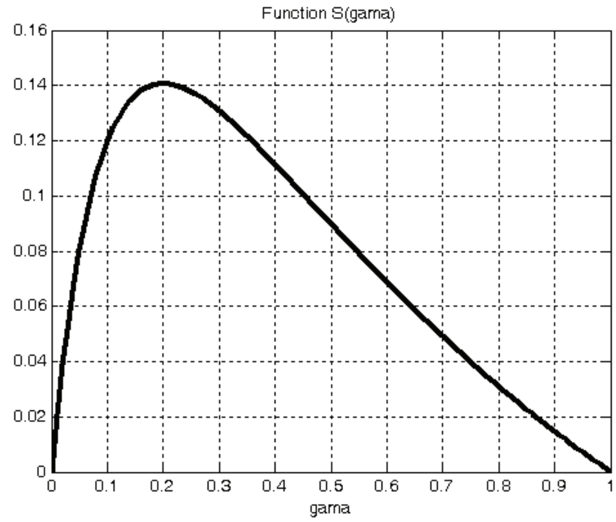

Fig. 5. Function $S(\gamma)$ *for* $d=2, \beta=1, L=1.5$ *and* $\tilde{a}=3/4$

In this case the maximum of the function $S(\gamma)$ is reched (see figure 5) for $\gamma_{opt} = 0.2$ and $S(\gamma_{opt}) = 0.14$. ■

5. FINAL REMARKS

Original analysis of the coefficient $\tilde{a}$ of the homogenized system (8) is presented using mathematical foundations of homogenization of periodic media (Cioranescu & Donato, 1999) for one-dimensional (N = 1) boundary value problem (1). Analitical formula of the coefficient $\tilde{a}$ was obtained thanks to acceptance of basic function (2) describing properties of a micrsostructure. Similar analitical formulas can be obtained for others basic functions, like for example in the figure 6 (homogenization of three-component's medium) or others allowing calcultion of the integral in formula (9).

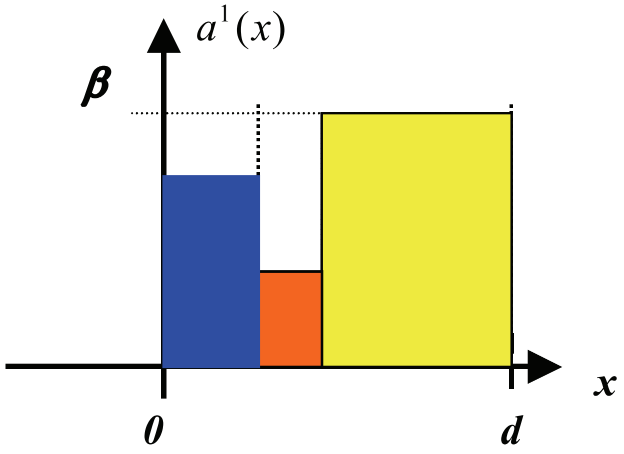

Fig. 6. *Basic function*

In the paper the original inverse problem is considered. It consists in determining of basic function this way to obtain given coefficient in result of homogenization. Solution of peresented inverse problem is not unique solution. In order to obtain unique solution suitable optimization problem is proposed (see example 3).

Homogenization of layered materials for $N < \infty$ (N-dimensional elliptic systems) was considered in the book (Cioranescu & Donato, 1999, p. 99, 106). In this case homogenized system is characterized by $N \times N$ constant matrix and the inverse problem can be studied, but it is more complicated. Generally important is that the complexity of any calculations depends on the domain in R^N, where given boundary value problem is considered. For example in figure 1 $N = 2$ and the domain is a square.

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HOMOGENIZACJA JEDNOWYMIAROWEGO SYSTEMU ELIPTYCZNEGO

Streszczenie

W pracy przedstawiono problem homogenizacji jednowymiarowych ośrodków o mikrostrukturze okresowej, dokładniej homogenizację jednowymiarowego systemu eliptycznego o warstwowej mikrostrukturze periodycznej. Podano przykład mikrostruktury okresowej, odpowiednio sparametryzowany, w którym można wyznaczyć parametry układu po homogenizacji. Następnie rozważono problem odwrotny: w jakich proporcjach należy mieszać składniki, by po homogenizacji uzyskać materiał o zadanych własnościach.

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